

Indexed sets

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Summation notation

“Recall” that we can write a long sum of a bunch of numbers using summation notation.

$$a_1 + a_2 + \cdots + a_n = \sum_{i=1}^n a_i$$

We can even write infinite sums:

$$1 + \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^i} + \cdots = \sum_{i=1}^{\infty} \frac{1}{2^i}$$

Indexed sets

Suppose we have a bunch of sets $A_1, A_2, \dots, A_n$. Then we can write:

$$A_1 \cup A_2 \cup \dots \cup A_n = \bigcup_{i=1}^n A_i$$

and

$$A_1 \cap A_2 \cap \dots \cap A_n = \bigcap_{i=1}^n A_i$$

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If $A_1, A_2, \dots, A_n$ are all sets, then

$$\bigcup_{i=1}^n A_i = \{x : x \text{ belongs to at least one set } A_i\}$$

- ▶ $A_1 = \{1, 4, 10, 12\}$
- ▶ $A_2 = \{5, 12, 15\}$
- ▶ $A_3 = \{1, 4, 15, 35\}$

$$\{1, 4, 10, 12, 5, 15, 35\}$$

What is $\bigcup_{i=1}^3 A_i$?

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$$\bigcap_{i=1}^n A_i = \{x : x \text{ belongs to every set } A_i\}$$

- ▶ $A_1 = \{1, 4, 10, 12\}$
- ▶ $A_2 = \{5, 12, 15\}$
- ▶ $A_3 = \{1, 4, 15, 35\}$

disjoint

What is $\bigcap_{i=1}^3 A_i$? = $\emptyset$

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One can also take the union and intersection of infinitely many sets.

$$\bigcup_{i=1}^{\infty} A_i \text{ and } \bigcap_{i=1}^{\infty} A_i.$$

Example. For each $i \in \mathbb{N}$, let

$$A_i = \{-i, 0, i\}$$

$$\begin{aligned} A_1 &= \{-1, 0, 1\} \\ A_2 &= \{-2, 0, 2\} \\ A_3 &= \{-3, 0, 3\} \\ &\vdots \end{aligned}$$

What is $\bigcup_{i=1}^{\infty} A_i$ and $\bigcap_{i=1}^{\infty} A_i$?

$$\bigcup_{i=1}^{\infty} A_i = \mathbb{Z}, \quad \bigcap_{i=1}^{\infty} A_i = \{0\}$$

Index sets

Instead of numbering the sets, one can label them with elements of any set I called an index set.

$\bigcup_{i \in I} A_i$ is the set of elements that belong to *at least one* of the sets A_i .

$\bigcap_{i \in I} A_i$ is the set of elements that belong to *every one* of the sets A_i .

$$\begin{aligned} I &= \{ \text{blcks on campus} \} \\ A_i &= \{ \text{set of room \#s for rooms "in" } i \} \end{aligned}$$

Index sets example

Let C be the set of Counties in the state of Connecticut (there are 8 of these). For each county $c \in C$, let $T(c)$ be the set of Towns in that County.

For example, if c is Tolland County, then the elements of $T(c)$ are Andover, Bolton, Columbia, Coventry, Ellington, Hebron, Mansfield, Somers, Stafford, Tolland, Union, Vernon, and Willington.

What is $\bigcup_{c \in C} T(c)$?

$$\mathcal{T} = \{ \text{all towns} \} = \bigcup_{c \in C} \underbrace{T(c)}$$

$$\underset{8}{C} \times \mathcal{T}$$

Index sets

Let

$$\mathbb{R}_+ = \{r : r \in \mathbb{R}, r > 0\}.$$

For every real number $r \in \mathbb{R}_+$, let

$$A_r = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < r^2\}.$$

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What is $\bigcap_{r \in \mathbb{R}_+} A_r$?

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What is $\bigcup_{r \in \mathbb{R}_+} A_r$?

Example

What is $\bigcap_{i \in \mathbb{N}} [0, i + 1]$?

Example

Suppose that I and J are sets, that $J \neq \emptyset$, and that $I \subseteq J$. Is

$$\bigcap_{a \in I} A_a \subseteq \bigcap_{a \in J} A_a?$$

Explain.